

Shapley's Game Value Axiomatic with restricted symmetry

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Abstract

According to a Shapley's result, there exists a unique game-value of cooperative games that satisfy axioms on additivity, efficiency, null-player property and symmetry. The original setting requires the symmetry with respect to arbitrary permutations of the players. We show that if we weaken the symmetry axiom to a symmetry with respect to a subgroup G of the permutation group S_n , the uniqueness of the game-value is satisfied if and only if the group G satisfies the following “supertransitivity” property:

For each subset $A \subset \{1, 2, \dots, n\}$, the stabilizer G_A of A acts transitively on A .

We classify all permutation groups satisfying this property and show that these are exactly the full symmetry groups S_n , alternating group A_n for $n \geq 4$ and any of the exotic embeddings of S_5 into S_6 for $n = 6$.

For a general subgroup G of S_n , the set of all game-values satisfying the axioms above with the symmetry axiom weakened to G -symmetry is an affine space of dimension $(\frac{dZ}{dx_1} - Z)|_{(2,2,\dots,2)} + 1$, where Z is the cycleindicator of the group G .

1 Basic definitions

1.1 Cooperative games

The following definitions can be found for example in [5].

Definition The pair (Ω, v) is a **cooperative game** (in characteristic function form) if Ω is a finite set of players and $v : 2^\Omega \rightarrow \mathbb{R}$ is a characteristic function that assigns to every coalition $R \subseteq \Omega$ an attainable profit $v(R)$ such that $v(\emptyset) = 0$.

A cooperative game is called **additive**, if for all $T, R \in 2^\Omega$ with $R \cap T = \emptyset$, $v(R \cup T) = v(R) + v(T)$.

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A cooperative game is called **finite**, if $|\Omega| < \infty$.

We denote by Γ the set of all finite cooperative games and by Γ_1 the set of all additive finite cooperative games.

Definition A **value** of games is an operator $\psi : \Gamma \rightarrow \Gamma_1$ s.t. $\Psi : (\Omega, v) \mapsto (\Omega, \psi \circ v)$ (the set Ω remains unchanged).

In particular, we define $\psi_i(v) := \psi \circ v(\{i\})$. Clearly, $\psi \circ v$ is uniquely determined by the numbers $\psi_i(v)$.

The **Shapley value** is a value ϕ defined by the formula

$$\phi_i \circ v = \sum_{R \ni \{i\}} \frac{\Delta_v(R)}{|R|}$$

where $\Delta_v(R) \in \mathbb{R}$ is a **Harsanyi dividend** of the coalition $R \subseteq \Omega$ defined by

$$\Delta_v(R) = \sum_{T \subseteq R} (-1)^{|R|-|T|} v(T)$$

Definition The unanimity basis is the basis $\{u_R\}_{\emptyset \neq R \subseteq \Omega}$ of the vector space of all cooperative games over the set Ω defined by

$$u_R(S) = \begin{cases} 1 & \text{if } R \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

1.2 Group theory

We say that G acts on the set $X = \{1, 2, \dots, n\}$, if G is a subgroup of the group S_X of permutations of X . The action of G on X is **transitive**, if for each $x, y \in X$, there exists a $g \in G$ such that $g \cdot x = y$. The stabilizer of a subset $A \subseteq X$ is the subgroup G_A of all elements $g \in G$ such that $g \cdot A \subseteq A$.

The next definition is not a standard one, but describes a property of permutation groups we will use later.

Definition Let $G \subset S_n$ be a subgroup of the symmetric group S_n acting. We say that G has a **supertransitive** action on $\{1, 2, \dots, n\}$ if the stabilizer G_A of any subset $A \subseteq \{1, 2, \dots, n\}$ acts transitively on A .

For any n , S_{n-1} may be imbedded into S_n as a set of transformation preserving one point. However, for $n = 6$, there exists an *exotic embedding* of S_5 into S_6 different from the standard one. It is well known that such an embedding is only one up to conjugation by an element of S_6 [2].

2 Introduction

The definition of a cooperative game as a point in \mathbb{R}^{2^Ω} illustrates an economic situation where a coalition profit depends in general on the involved players

in a non-additive way (usually, a more general property called super-additivity is required). These ideas were developed in fifties by Neumann and Morgenstern. [13] Values of games provide a tool how to evaluate the contributions of the players. In particular, the Shapley value describes a way how to do it in a “fair” way. Formally, it is a decomposition of the game v in the *unanimity basis*.

An alternate ways leading to Shapley value are game-value axiomatics. There exists a couple of axiomatic systems on game-values such that the Shapley value is their only solution: the original Shapley’s axiomatics [11], and simplest version [1], Young’s axiomatics [14], van den Brink’s axiomatization [12], Kar’s axiomatization [6].

Shapley theorem [9, 11] proves the existence of a unique game-value operator ϕ assuming it satisfies the following four axioms:

- (1) Linearity: ¹ $\phi(\alpha v + \beta v') = \alpha \phi(v) + \beta \phi(v')$ for all $(\Omega, v), (\Omega, v') \in \Gamma$ and $\alpha, \beta \in \mathbb{R}$.
- (2) Null-player property: if $i \in \Omega$ is a “null-player”, i.e. $\forall R \subseteq \Omega \ v(R \cup \{i\}) = v(R)$, then $\phi_i(v) = 0$.
- (3) Efficiency: For all games (Ω, v) $\sum_i \phi_i(v) = v(\Omega)$.
- (4) Symmetry (sometimes called *anonymity*): $\phi(\rho(i))(\rho \cdot v) = \phi_i(v)$ for every permutation $\rho \in S_\Omega$ (the function $\rho \cdot v$ is defined by $\rho \cdot v(\rho(R)) := v(R)$ for any $R \subseteq \Omega$).

Axioms (1) – (4) are independent. Gilles [5] and Schmeidler [3] give examples of values satisfying any 3 of them and not the 4th.

In the original economic interpretation, the fourth axiom (Symmetry) is an expression of equality of all the participating players. This means that the game-value assigned to them is calculated only from their contributions to the coalitions and does not depend on the particular identity of the player. In this paper, we focus on the consequences of weakening of the Symmetry axiom. We describe the consequences for uniqueness if there is not a complete equality of the players (i.e. with respect to each permutation $\rho \in S_\Omega$) but if the symmetry is satisfied only for permutations ρ from a subset (in particular a subgroup) $G \subseteq S_\Omega$. We will investigate the axiomatics consisting of axioms (1) – (3) and (4) G -symmetry: $\phi_{\rho(i)}(\rho \cdot v) = \phi_i(v)$ for every permutation $\rho \in G$.

Example 0. $G = \{Id\}$. The fourth axiom implies invariance only with respect to the identity. Here the requirement of G -symmetry is always satisfied and the axioms are reduced to axioms (1) – (3).

The following construction is taken from Gilles [5].

Consider a function $\gamma : 2^\Omega \rightarrow \Omega$ with $\gamma(R) \in R$ for all $\emptyset \neq R \subseteq \Omega$. Now we

¹In the original version there is a formally weaker “additivity” condition $\phi(v + v') = \phi(v) + \phi(v')$ for all $(\Omega, v), (\Omega, v') \in \Gamma$. Assuming the axiom of choice, there exist other then linear solution of the functional equation $f(x + y) = f(x) + f(y)$, however, they are all eliminated by a further reasonable condition on the solution. Linearity is a consequence of the additivity property + continuity, additivity + boundedness on compact sets, additivity + measurability etc.

All the proofs in this paper are valid independent of the form of this axiom.

define the value ψ as the allocation

$$\psi_i(v) = \sum_{i=\gamma(R)} \Delta_v(R)$$

This value satisfies efficiency, null player property as well as linearity. Regarding the null-player property we remark that any player $i \in R$ is not a null-player if $\Delta_v(R) \neq 0$.

Example 1. (“Caste system”) The set Ω is split into k nonempty disjoint subsets (“castes”) $\Omega_1, \dots, \Omega_k$ and G is chosen so that it guarantees equality within each Ω_i . Formally, $G = \{\pi \in S_\Omega; \forall i \pi(\Omega_i) = \Omega_i\}$.

For a particular game v and caste r we define the game v_r by $v_r(R) = 0$ for R disjoint from Ω_r and $v_r(R) = v(R \cup (\Omega - \Omega_r))$ otherwise. Further we define φ^r as the Shapley-value of the game v_r . In v_r , the nullplayers are all players who are not members of Ω_r and players in Ω_r who were nullplayers already in game v . Axioms (1) – (3) are satisfied for each affine combination of φ^r , $r \in \{1, \dots, k\}$.

Here the symmetry axiom did not guarantee the invariance of the value with respect to symmetries permuting players between castes and it was possible to choose any caste and distribute its coalition profit only within the caste. Equality between players was not satisfied.

Less obvious is the result that for a unique determination of the value, the transitivity of the symmetry group G is not a sufficient condition for $|\Omega| \geq 3$ and for $|\Omega| \geq 4$ even the possibility to interchange any two pairs of players does not imply uniqueness of value.

Example 2. (Cyclic group) For $\Omega = \{1, 2, 3\}$ we choose $G = C(\Omega) = \{\pi_k; k \in \{1, 2, 3\}\}$ where $\pi_k : i \mapsto (i + k) \bmod k$.

We decompose a general game v over Ω into the unanimity basis $\{u_S\}$:

$$v = \alpha_{\{0\}} u_{\{0\}} + \alpha_{\{1\}} u_{\{1\}} + \alpha_{\{2\}} u_{\{2\}} + \alpha_{\{0,1\}} u_{\{0,1\}} + \alpha_{\{1,2\}} u_{\{1,2\}} + \alpha_{\{0,2\}} u_{\{0,2\}} + \alpha_{\{0,1,2\}} u_{\{0,1,2\}}$$

and define for $\lambda \in \mathbb{R}$

$$\begin{aligned} \varphi_0^\lambda(v) &= \alpha_{\{0\}} + \frac{\alpha_{\{0,1,2\}}}{3} + \lambda \alpha_{\{0,1\}} + (1 - \lambda) \alpha_{\{0,2\}} \\ \varphi_1^\lambda(v) &= \alpha_{\{1\}} + \frac{\alpha_{\{0,1,2\}}}{3} + \lambda \alpha_{\{1,2\}} + (1 - \lambda) \alpha_{\{0,1\}} \\ \varphi_2^\lambda(v) &= \alpha_{\{2\}} + \frac{\alpha_{\{0,1,2\}}}{3} + \lambda \alpha_{\{0,2\}} + (1 - \lambda) \alpha_{\{1,2\}} \end{aligned}$$

Each of the values φ^λ satisfies axioms (1) – (3) and the C_Ω -symmetry, but only for $\lambda = 1/2$ it is the Shapley value.

For a more general case $\Omega = \{0, 1, \dots, n-1\}$ and C_Ω we may define φ^λ symmetric with respect to C_Ω for instance by the formula

$$\varphi_i^\lambda(v) = \varphi_i(v) + \lambda \alpha_{\Omega - \{i+1 \bmod(n)\}} - \lambda \alpha_{\Omega - \{i-1 \bmod(n)\}}$$

where $v = \sum_{\emptyset \neq R \subseteq \Omega} \alpha_R u_R$, φ is the Shapley value and $\lambda \in \mathbb{R}$. For $n > 3$ this construction does not exhaust all possible cases.

Example 3. (Dihedral group) For $n \geq 4$ and $\Omega = \{0, 1, \dots, n-1\}$, choose D_Ω to be the dihedral group $\{\pi_{k\pm}; k \in \{0, 1, \dots, n-1\}\}$, whereas the permutation $\pi_{i\pm} : i \mapsto (\pm i + k) \bmod n$. For $\lambda \in \mathbb{R}$ we define

$$\varphi_i^\lambda(v) = \varphi^\lambda(v) + \lambda \alpha_{\{i-1, i, i+1\} \bmod n} - \frac{\lambda}{2} \alpha_{\{i, i+1, i+2\} \bmod n} - \frac{\lambda}{2} \alpha_{\{i, i-1, i-2\} \bmod n}.$$

φ^λ satisfies axioms (1) – (3) and the D_Ω -symmetry. D_Ω implies the equality of players in the sense that for any two players (i, j) there exists a permutation interchanging them.

However, we will show that in general, we don't have to require the symmetry of the value with respect to *all* permutations in order to obtain a unique value. For $|\Omega| \geq 4$ there exists a nontrivial subgroup G of S_Ω such that axioms (1) – (3) and the G -symmetry implies the uniqueness of value. In the special case of $|\Omega| = 6$, there exist 7 such subgroups.

3 Symmetry and uniqueness of game-value

In Section 1.2, we defined a group $G \subset S_\Omega$ of permutation to be supertransitive, if the stabilizer G_A acts transitively on A for each subset $A \subseteq \Omega$. In other words, if for each A and each $x, y \in A$, there exists a $g \in G$ such that $g(A) = A$ and $g \cdot x = y$.

For $|\Omega| > 2$, the supertransitivity of $G \subset S_\Omega$ is a stronger condition than transitivity. For example, the cyclic group $C_\Omega \subset S_\Omega$ acts transitively on Ω but not supertransitively.

Theorem 3.1 *Let Ω be finite and $G \leq S_\Omega$.*

- (a) *There exists a unique game-value operator over Ω satisfying axioms (1) – (3) and G -symmetry if and only if G acts supertransitively on Ω .*
- (b) *There exists a unique game-value operator over Ω satisfying axioms (1) – (3) and G -symmetry if and only if one of the following conditions is satisfied:*

- $G = S_\Omega$, the full symmetric group
- $G = A_\Omega$, the alternating group
- $|\Omega| = 6$ and $G \simeq S_5$ corresponds to the exotic embedding $S_5 \hookrightarrow S_6$

Proof Let Ω be finite with $|\Omega| = n$. Game-values over Ω satisfying the linearity condition are exactly all linear mappings $\mathbb{R}^{2^\Omega - \{\emptyset\}} \rightarrow \mathbb{R}^n$. In the first space, we choose the unanimity basis and represent the game-values as matrices of size $n \times (2^n - 1)$

It follows from the original Shapley result that for the whole group $G = S_\Omega$ there exists a unique matrix $Sh = A_{S_\Omega}$ satisfying axioms (1) – (3) and G -symmetry. In particular,

$$Sh = (a_{iR})_{i \in \Omega}^{R \in 2^\Omega - \{\emptyset\}} : a_{iR} = \begin{cases} 0 & \text{if } i \notin R \\ \frac{1}{|R|} & \text{if } i \in R \end{cases}$$

This game-value over Ω satisfies axioms (1) – (3) and the symmetry axiom for S_Ω as well as for any smaller set of permutations.

For a subgroup $G \leq S_\Omega$, let us analyse conditions on the matrix elements of A_G such that the axioms (including G -symmetry) are satisfied.

i) A_G corresponds to a game-value satisfying the null-player-property iff $a_{iR} = 0$ for all couples (i, R) such that $i \notin R$. Validity of this condition follows directly from applying the null-player-property on games u_R forming the unanimity basis and extending this linearly on all games.

ii) A_G corresponds to a game-value satisfying the efficiency axiom iff for all $R \in 2^\Omega - \{\emptyset\}$, $\sum_{i \in R} a_{iR} = 1$. Again, this follows from the applying the efficiency condition on games forming the unanimity basis and linearity.

iii) We define an action of the group G on the set $X_0 := \{(i, R); i \in \Omega, \emptyset \neq R \subseteq \Omega\}$ and $X = \{(i, R); i \in R \subseteq \Omega\}$ in a natural way:

$$g \cdot (i, R) := (g(i), g(R)).$$

Applying the G -symmetry condition on a game u_R from the unanimity basis implies that for each $g \in G$, $a_{i,R} = a_{g(i),g(R)}$. So, a *necessary condition* for the G -symmetry is the condition that elements of A_G are constant on the G -orbits of X .

If we decompose a general game v into the unanimity basis

$$v = \sum_{\emptyset \neq R \subseteq \Omega} \alpha_R u_R,$$

then for any $\pi \in G$

$$\pi(v) = \sum_{\emptyset \neq R \subseteq \Omega} \alpha_R u_{\pi(R)} = \sum_{\emptyset \neq R' = \pi(R) \subseteq \Omega} \alpha_{\pi^{-1}(R')} u_{R'}$$

and assuming that the values a_{iR} are constant on the G -orbits, we have

$$(A_G(\pi(v)))|_{\pi(i)} = \sum_{\emptyset \neq R \subseteq \Omega} a_{\pi(i)R} \alpha_{\pi^{-1}(R)} = \sum_{\emptyset \neq R \subseteq \Omega} a_{\pi(i),\pi(R)} \alpha_R = \sum_{\emptyset \neq R \subseteq \Omega} a_{i,R} \alpha_R = (A_G v)|_i,$$

so the condition on constant G -orbits is also sufficient for the efficiency.

So, a matrix representing a linear operator $\mathbb{R}^{2^\Omega - \{\emptyset\}} \rightarrow \mathbb{R}^\Omega$ in the unanimity basis of the first space and a canonical basis of the second space, represents a game-value operator satisfying axioms (1) – (3) and the G -symmetry axiom if and only if the following conditions are satisfied:

- $a_{iR} = 0$ if $i \notin R$
- The sum of elements in each column is 1
- The matrix elements are constant on the orbits of the G -action.

If the action of G on Ω is supertransitive, then there exists for each $\emptyset \neq R \subseteq \Omega$ and each $i, j \in R$ an element $g \in G$ such that $g(i) = j$ and $g(R) = R$. For a

particular $\emptyset \neq R \subseteq \Omega$, all elements $\{(i, R); i \in R\}$ lie on the same orbit of G , so all these elements are equal. Formally, for all $R \subset \Omega$ and $i, j \in R$, $a_{i,R} = a_{j,R}$. The null-player property implies that $a_{i,R} = 0$ for $i \notin R$ and together with the efficiency condition we obtain the condition that for each $i \in R$, $a_{i,R} = 1/|R|$. So, we proved uniqueness of the game-value and it has to be equal to the Shapley value.

On the other hand, if the action of G on Ω is not supertransitive, there exists a nonempty subset $\tilde{R} \subseteq \Omega$ such that the stabilizer $G_{\tilde{R}}$ has not a transitive action on \tilde{R} . In this case, there exists an R intersecting more than one G -orbit. In the corresponding matrix column we may choose $a_{i,R} = 0$ if $i \notin R$ and the other elements $a_{j,R}$ arbitrary, constant on G -orbits and such that $\sum_j a_{j,R} = 1$. For pairs (i', R') on the G -orbit of (i, R) , we define $a_{\pi(i), \pi(R)} = a_{i,R}$. Remaining elements may be equal to elements of the original Shapley matrix. In this way, we may obtain an infinite number of different game-values which proves the non-uniqueness of the game-value if G does not act supertransitively on Ω . We proved the part (a) of the Theorem.

For part (b), it remains to prove that the groups listed in the Theorem are exactly the groups acting supertransitively on $\{1, \dots, n\}$. The proof of this can be found in [7].

4 Symmetry and non-uniqueness

For a finite fixed set Ω and an arbitrary set $H \subseteq S_\Omega$ of permutations (not necessary a subgroup) we denote by $\mathcal{A}_H \subseteq \mathbb{R}^{(2^\Omega - \{\emptyset\}) \times \Omega}$ the set of operators satisfying axioms (1) – (3) and π -symmetry with respect to any $\pi \in H$.

Theorem 4.1 *For each $H \subseteq S_\Omega$, \mathcal{A}_H is a nonempty affine subspace of $\mathbb{R}^{(2^\Omega - \{\emptyset\}) \times \Omega}$ and $\mathcal{A} = \mathcal{A}_{G(H)}$ where $G(H)$ is the group generated by all permutations in H .*

Proof The axioms determine the matrix element of operators in \mathcal{A}_H to satisfy the equations

$$\begin{aligned} a_{i,R} &= 0 \quad \text{for } i \notin R \\ \sum_i a_{i,R} &= 1 \quad \text{for } \emptyset \neq R \subseteq \Omega \\ a_{i,R} - a_{\pi(i), \pi(R)} &= 0 \quad \text{for } \pi \in H. \end{aligned}$$

If elements of a matrix A satisfies these conditions, the last one for permutations π and ρ , then it satisfies it also for the permutation $\pi\rho$ and π^{-1} . Nonemptiness of the solution set follows from the fact that the Shapley value (symmetric with respect to all permutations) clearly belongs to the set \mathcal{A}_H .

Theorem 4.2 *Let G be a subgroup of S_Ω . Let $X = \{(i, R); i \in R \subseteq \Omega\}$, $\chi = \{R; \emptyset \neq R \subseteq \Omega\}$ and let G act on the sets X and χ , extending naturally its action on Ω . Then \mathcal{A}_G is an affine subspace of $\mathbb{R}^{(2^\Omega - \{\emptyset\}) \times \Omega}$ and its dimension is*

- (a) $\dim = \#(G\text{-orbits in } X) - \#(G\text{-orbits in } \chi)$
(b) $\dim = \frac{1}{|G|} (\sum_{\pi \in G} (\# \text{fixedpoints}(\pi) * 2^{\# \text{cycles}(\pi) - 1}) - \sum_{\pi \in G} 2^{\# \text{cycles}(\pi)}) + 1$
(c) $\dim = (\frac{dZ_G}{dx_1} - Z_G)|_{(2,2,\dots,2)} + 1$
where Z_G is the cycle indicator ("Zyklenzeiger") of the group G [4, p. 85]

$$Z_G(x_1 \dots x_n) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{j_1(\pi)} \dots x_n^{j_n(\pi)}$$

where $j_k(\pi)$ denotes the number of cycles of length k in the permutation π .

Proof We will call the G -orbits of X "orbits" and the G -orbits of χ "metaorbits". Elements of \mathcal{A}_G correspond to matrices (a_{iR}) such that $a_{iR} = 0$ if $i \notin R$, $\sum_i a_{iR} = 1$ for each R and a_{iR} are constant on the orbits.

Let $p : X \rightarrow \chi$ by the natural projection. Then for any $x := (i, R) \in X$ and $g \in G$, the projection $p(gx) \in g(p(x))$, so an orbit is a refinement of a metaorbit. For any $R \subseteq \Omega$, the stabilizer G_R acts on R and R splits into k_R orbits $\{R_1, \dots, R_{k_R}\}$ with respect to this action. If $R' = gR$ is on the same metaorbit, then the stabilizer of R' is gG_Rg^{-1} and g maps any G_R -orbit $R_i \subseteq R$ bijectively onto a $G_{R'}$ -orbit $R'_i \subseteq R'$. So, $k_R = k_{R'}$ and $|R_i| = |R'_i|$ for $i = 1, \dots, k_R$. For a meta-orbit m , we define $k_m := k_R$ for any $R \in m$ and $l_{mi} = |R_i|$ for $i = 1, \dots, k_m$. We see that these numbers are independent of the choice of R .

We will say that a metaorbit m contains an orbit Gx , if $p(x) \in m$. A metaorbit m contains k_m orbits.

For any metaorbit $m \in \chi/G$ containing the orbits $\{o_1, \dots, o_{k_m}\}$, we may choose real numbers c_{mi} such that $\sum_{i=1}^{k_m} c_{mi} l_{mi} = 1$ with $k_m - 1$ degrees of freedom. Choosing such numbers c_{mi} for all metaorbits m gives

$$\sum_{m \in M} (k_m - 1) = \sum_{m \in M} k_m - |\chi/G| = |X/G| - |\chi/G|$$

degrees of freedom. Any such choice of c_{mi} defines a matrix of game-value operator given by

$$a_{iR} = \begin{cases} c_{mi} & \text{if } i \in R_i \subseteq R \in m \\ 0 & \text{if } i \notin R \end{cases}$$

and such game-value satisfies axioms (1) – (3) and G -symmetry. The number of degrees of freedom for the choice of c_{mi} is equal to the dimension of \mathcal{A}_G . This proves part (a).

(b) Burnside lemma [10, p. 58] enables to express the number of orbit of a group action. If a finite group H acts on a finite set Y , then

$$|Y/H| = |H| \sum_{h \in H} |\{y \in Y; h(y) = y\}|. \quad (1)$$

A permutation $\pi \in G$ fixes those sets $R \subseteq \Omega$ that don't split any cycle of π . There exists $2^{\# \text{cycles}(\pi)}$ such sets, $2^{\# \text{cycles}(\pi)} - 1$ of them nonempty. So,

$$|\chi/G| = \left(\frac{1}{|G|} \sum_{\pi} 2^{\# \text{cycles}(\pi)}\right) - 1.$$

Elements of X fixed by π are pairs (i, R) such that $i \in R$, $\pi(i) = i$ and $\pi(R) = R$. There exists $\# \text{fixedpoints}(\pi) * 2^{\# \text{cycles}(\pi)-1}$ such pairs and the dimension of \mathcal{A}_G is equal to their difference. This proves part (b). Statement (c) follows from (b) by direct computation.

4.1 Examples

We calculate now the dimensions of the space of game-values in the cases when G is a trivial group, for the Caste system, for the Cyclic and Dihedral groups.

Example 0 (continuation). For the trivial group $G = \{Id\}$, the cycle polynomial is $Z(x_1) = x_1^n$ and $\dim \mathcal{A}_G = n2^{n-1} - 2^n + 1$.

Example 1 (continuation). For the Caste system, if Ω is a disjoint union of subsets Ω_r and the symmetry group G is a product of the symmetric groups S_{Ω_r} . The cycle indicator is $Z_G = \prod_{r=1}^k Z_{S_{\Omega_r}}$.

We know from the proof of the last Theorem that the number of metaorbits is $|\chi/G| = \frac{1}{|G|} \sum_g 2^{\# \text{cycles}(g)}$. In particular, for $G = S_n$, $|\chi/G| = n + 1$, because metaorbits of S_n are $O_s = \{R \subseteq \Omega; |R| = s\}$ for $s = 0, 1, \dots, n$. This enables us to calculate

$$Z_{S_n}|_{(2, \dots, 2)} = \frac{1}{n!} \sum_{\pi} 2^{j_1 + \dots + j_n} = \frac{1}{n!} \sum_{\pi} 2^{\# \text{cycles}(\pi)} = |\chi/S_n| = n + 1.$$

If $G = S_n$, then the Shapley value is the only game value, so it follows from Theorem 4.2 that $(\frac{dZ_{S_n}}{dx_1} - Z_{S_n})|_{(2, \dots, 2)} + 1 = 0$ and $\frac{dZ_{S_n}}{dx_1}|_{(2, \dots, 2)} = n$. So, for $G = \prod_{r=1}^k S_{\Omega_r}$

$$\frac{dZ_G}{dx_1}|_{(2, 2, \dots, 2)} = \left(\sum_{r=1}^k \frac{dZ_{S_{\Omega_r}}}{dx_1} \prod_{s \neq r} Z_{S_{\Omega_s}} \right)|_{(2, 2, \dots, 2)} = \sum_{r=1}^k |\Omega_r| \prod_{s \neq r} (1 + |\Omega_s|)$$

and

$$\dim \mathcal{A}_G = \left(\sum_{r=1}^k \frac{|\Omega_r|}{1 + |\Omega_r|} - 1 \right) \prod_{r=1}^k (1 + |\Omega_r|) + 1.$$

For the case of two castes $k = 2$ this simplifies to $|\Omega_1| \times |\Omega_2|$.

Example 2 (continuation). If G is the cyclic group $C_n \subseteq S_n$, the cycle indicator is $Z_{C_n} = \frac{1}{n} \sum_{f|n} \phi(f) x_f^{n/f}$, where $\phi(f)$ is the Euler totient function $\phi(f) = p_1^{k_1-1}(p_1-1) \dots p_r^{k_r-1}(p_r-1)$, where $f = p_1^{k_1} \dots p_r^{k_r}$ is the prime number decomposition.[4, p. 86]. Substituting into the formula (c) of Theorem 4.2 gives

$$\dim \mathcal{A}_G = 2^{n-1} - \frac{1}{n} \sum_{f|n} \phi(f) 2^{n/f} + 1.$$

Example 3 (continuation). If G is the Dihedral group $D_n \subseteq S_n$, the cycle indicator is [8]

$$Z_{D_n} = \frac{1}{2} Z_{C_n} + \begin{cases} \frac{1}{2} x_1 x_2^{(n-1)/2} & \text{for } n \text{ odd} \\ \frac{1}{4} (x_1^2 x_2^{(n-2)/2} + x_2^{n/2}) & \text{for } n \text{ even} \end{cases}$$

and hence the dimension

$$\dim \mathcal{A}_{D_n} = \frac{1}{2} \dim(\mathcal{A}_{C_n}) - \begin{cases} 2^{(n-3)/2} & \text{for } n \text{ odd} \\ 2^{(n-4)/2} & \text{for } n \text{ even} \end{cases} + \frac{1}{2}$$

5 Appendix

The original proof [11] uses the symmetry axiom only for showing that in the game Cu_R (C is a constant) each member of the coalition R will obtain the same amount, because for any two players $i, j \in R$ there exists a permutation π satisfying $\pi(i) = j$ and $\pi(R) = R$. It follows from efficiency and zero player property that the share of each player in R will be $C/|R|$. The linearity axiom extend uniqueness to all games. However, the only requirement for symmetry were supertransitivity of the symmetry group, so we see immediately that this requirement is also sufficient.

Similarly, Youngs axiomatization derives the Shapley value as the only game-value operator satisfying axioms of efficiency, symmetry and *strong monotonicity*. [14]

Definition A value φ satisfies strong monotonicity if for all games v, w

$$D_i v(R) \geq D_i w(R) \text{ for all } R \subseteq \Omega \text{ implies } \varphi_i(v) \geq \varphi_i(w)$$

where

$$D_i v(R) = \begin{cases} v(R) - v(R - \{i\}) & \text{if } i \in R \\ v(R \cup \{i\}) - v(R) & \text{if } i \notin R \end{cases}$$

denotes the marginal contribution of a player to an arbitrary coalition $R \subseteq \Omega$.

In the uniqueness proof Young also uses only the supertransitivity property of the symmetry group.

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